Quaternion Exercises (Computer Animation)

1. What is the quaternion $\mathbf{q}_1$ that represents the rotation of 180 degree about the x-axis?

2. What is the quaternion $\mathbf{q}_2$ that represents the rotation of 180 degree about the z-axis?

3. What rotation is represented by composite quaternion $\mathbf{q} = \mathbf{q}_1\mathbf{q}_2$? Answer by specifying its rotation angle and axis.

4. Let $\mathbf{x}$ be a point and let $\mathbf{X} = (0, \mathbf{x})$ be a quaternion whose scalar part is zero and whose vector part is equal to $\mathbf{x}$. Show that if $\mathbf{q} = (w, \mathbf{v})$ is a unit quaternion, the product $\mathbf{q}\mathbf{X}\mathbf{q}^{-1}$ is a purely imaginary quaternion and the vector part of $\mathbf{q}\mathbf{X}\mathbf{q}^{-1}$ satisfies

$$(w^2 - \mathbf{v} \cdot \mathbf{v})\mathbf{x} + 2\Big(w(\mathbf{v} \times \mathbf{x}) + (\mathbf{x} \cdot \mathbf{v})\mathbf{v}\Big). \tag{1}$$

Note that the vector part describes the point to which $\mathbf{x}$ is rotated under the rotation associated with $\mathbf{q}$.

5. Show that $\mathbf{q}$ and $-\mathbf{q}$ represent the same rotation using the result of Exercise 4.

6. Compare the number of additions and multiplications needed to perform the following operations:

- Compose two rotation matrices.
- Compose two quaternions.
- Apply a rotation matrix to a vector.
- Apply a quaternion to a vector (as in Exercise 4).

Count a subtraction as an addition, and a division as a multiplication.

7. Show that a rigid body rotating at angular velocity $\omega(t) \in \mathbb{R}^3$ can be represented by the quaternion differential equations

$$\dot{\mathbf{q}}(t) = \frac{1}{2}\Big(0, \omega(t)\Big)\mathbf{q}(t). \tag{2}$$

Hint: Recall that the angular velocity $\omega(t)$ indicates that the body is instantaneously rotating about the ω axis with magnitude $\|\omega(t)\|$. Suppose that a body were to rotate with a constant angular velocity $\omega(t)$. Then the rotation of the body after a period of time Δt is represented by the quaternion

$$\left(\cos\frac{\|\omega(t)\|\Delta t}{2}, \frac{\omega(t)}{\|\omega(t)\|}\sin\frac{\|\omega(t)\|\Delta t}{2}\right). \tag{3}$$

At times $t+\Delta t$ (for small Δt), the orientation of the body is (to within the first order)

$$\mathbf{q}(t+\Delta t) = \left(\cos\frac{\|\omega(t)\|\Delta t}{2}, \frac{\omega(t)}{\|\omega(t)\|}\sin\frac{\|\omega(t)\|\Delta t}{2}\right)\mathbf{q}(t). \tag{4}$$

Compute $\dot{\mathbf{q}}$ by differentiating the above equation.