Kinematics

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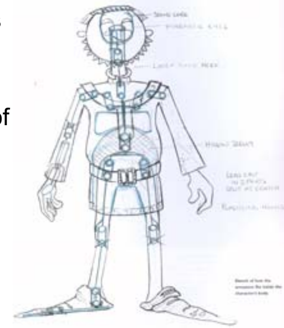
Kinematics

- How to animate skeletons (articulated figures)
- ***Kinematics*** is the study of motion without regard to the forces that caused it

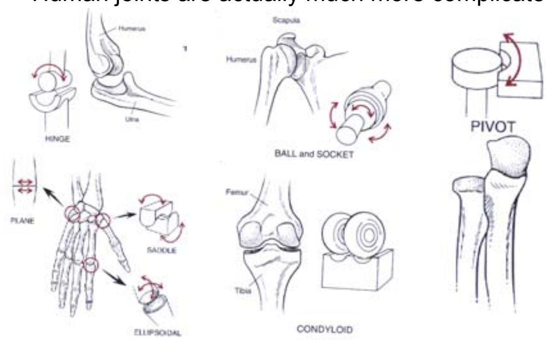


Hierarchical Models

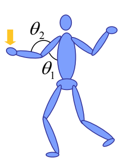
- Tree structure of joints and links
  - The root link can be chosen arbitrarily
- Joints
  - Revolute (hinge) joint allows rotation about a fixed axis
  - Prismatic joint allows translation along a line
  - Ball-and-socket joint allows rotation about an arbitrary axis

Human Joints

- Human joints are actually much more complicated

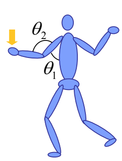


Forward and Inverse Kinematics



$$(\mathbf{p},\mathbf{q}) = \mathrm{F}(\theta_i)$$

Forward Kinematics



$$\theta_i = \mathrm{F}^{-1}(\mathbf{p},\mathbf{q})$$

Inverse Kinematics

3D Position and Orientation

- The position and orientation of an object is represented as a rigid transformation
- Vector & Quaternion

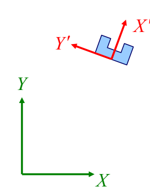
$$\mathbf{q}\mathbf{v}\mathbf{q}^{-1} + \mathbf{p}$$

- Vector & 3x3 Matrix

$$\mathbf{R}\mathbf{v} + \mathbf{p}$$

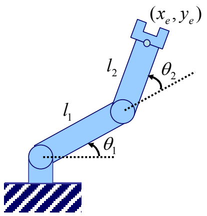
- 4x4 Matrix

$$\mathbf{T}\mathbf{v} = \begin{pmatrix} \mathbf{R} & \mathbf{p} \\ \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{v} \\ 1 \end{pmatrix}$$



Forward Kinematics: A Simple Example

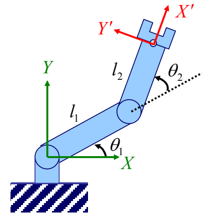
- A simple robot arm in 2-dimensional space
  - 2 revolute joints
  - Joint angles are known
  - Compute the position of the end-effector



$$x_e = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2)$$
$$y_e = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2)$$

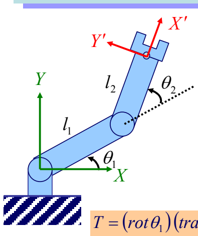
Forward Kinematics: A Simple Example

- Forward kinematics map as a coordinate transformation
  - The body local coordinate system of the end-effector was initially coincide with the global coordinate system
  - Forward kinematics map transforms the position and orientation of the end-effector according to joint angles



$$\begin{pmatrix} x_e \\ y_e \\ 1 \end{pmatrix} = \begin{pmatrix} & & \\ & T & \\ & & \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

A Chain of Transformations

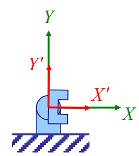


$$\begin{pmatrix} x_e \\ y_e \\ 1 \end{pmatrix} = \begin{pmatrix} & & \\ & T & \\ & & \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$T = (rot\,\theta_1)(transl_1)(rot\,\theta_2)(transl_2)$$
$$= \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & l_1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & l_2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thinking of Transformations

- In a view of body-attached coordinate system



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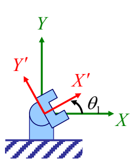

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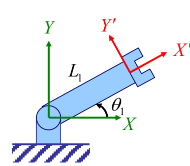

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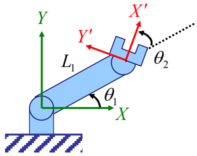

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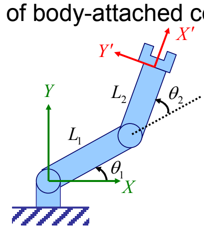

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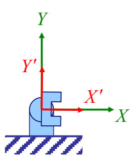
Thinking of Transformations

- In a view of global coordinate system



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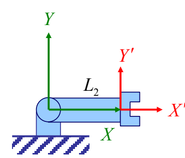

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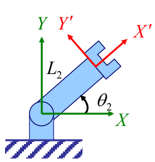

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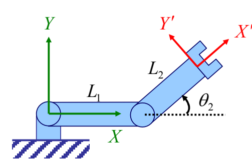

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Thinking of Transformations

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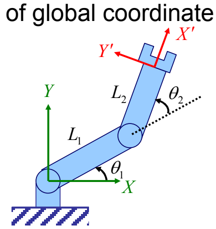


$$T = (rot\,\theta_1)(transl_1)(rot\,\theta_2)(transl_2)$$

$$= \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & l_1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & l_2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

How to Handle Ball-and-Socket Joints ?

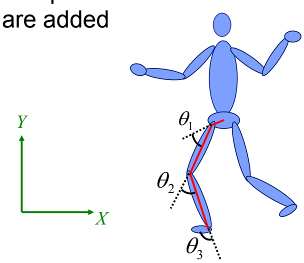
- Three revolute joints whose axes intersect at a point (equivalent to Euler angles), or
- 3D rotation about an arbitrary axis

$$T = (transl_1)(rot\theta_x)(rot\theta_y)(rot\theta_z)(transl_2)$$

$$= \cdots \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x & 0 \\ 0 & \sin\theta_x & \cos\theta_x & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta_y & 0 & -\sin\theta_y & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta_y & 0 & \cos\theta_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta_z & -\sin\theta_z & 0 & 0 \\ \sin\theta_z & \cos\theta_z & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdots$$

Floating Base

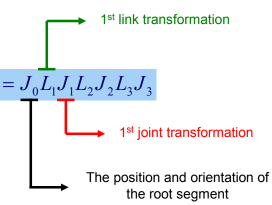
- The position and orientation of the root segment are added



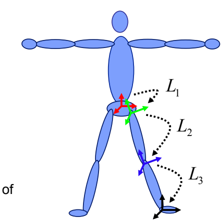
$$T = (rot\,\theta_r)(transl_r)(transl_0)(rot\,\theta_0)(transl_1)(rot\,\theta_1)(transl_2)(rot\,\theta_2)$$

Joint & Link Transformations

- Each segment has its own coordinate frame
- Forward kinematics map is an alternating multiple of
  - ***Joint transformations*** : represents joint movement
  - ***Link transformations*** : defines a frame relative to its parent



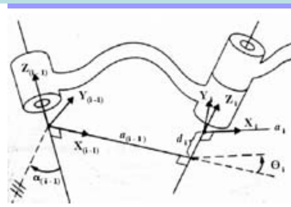
$$T = J_0 L_1 J_1 L_2 J_2 L_3 J_3$$



Joint & Link Transformations

- Both are rigid transformations in general
  - Joint transformations may include translation
    - Human joints are not ideal hinges
  - Link transformations may include rotation
    - Some links are twisted

Denavit-Hartenberg Notation



$$L_i = Rot(X, \alpha_{i-1}) \cdot Trans(X, a_{i-1}) \cdot Trans(Z, d_i) \cdot Rot(Z, \theta_i)$$

a_i = the distance from Z_i to Z_{i+1} measured along X_i

α_i = the angle between Z_i and Z_{i+1} measured about X_i

d_i = the distance from X_{i-1} to X_i measured along Z_i

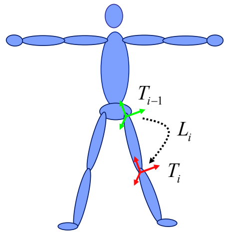
θ_i = the angle between X_{i-1} and X_i measured about Z_i

Link Transformations

- How do you compute the link transform for the i[th] joint if you know the position and orientation of the i[th] joint as well as its parent's position and orientation at the neutral pose ?

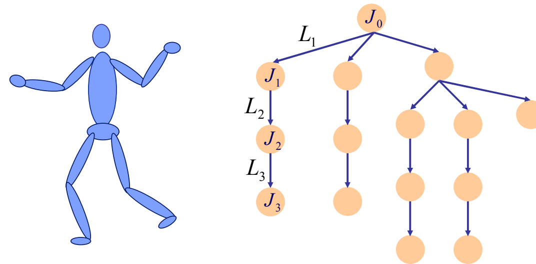
$$T_{i-1}L_i = T_i$$

$$L_i = T_{i-1}^{-1}T_i$$



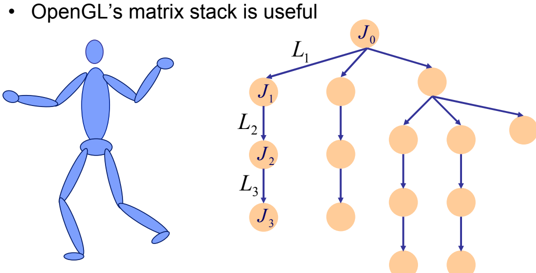
Representing Hierarchical Models

- A tree structure
  - A node contains a joint transformation
  - A arc contains a link transformation



Depth-First Tree Traversal

- Draw graphics needs to compute the position and orientation of all links
- OpenGL's matrix stack is useful



Summary

- Kinematics is the study of motion of articulated figures
  - Kinematics does not consider physics (forces, mass, …)
- Forward kinematics is straightforward
  - Forward kinematics map can be considered as a coordinate transformation
- The next lecture: How to solve the inverse problem