Chapter 2

Preliminary

Orientations as well as positions are important to describe motions. Quaternions, discovered by Sir William Rowan Hamilton [39], provide a solid base to represent the orientation of a 3-dimensional object. In this chapter, we give a brief introduction to quaternions to explain the geometric and algebraic structures of the unit quaternion space $\mathbb{S}^3$.

Quaternion Basics

The four-dimensional space of quaternions is spanned by a real axis and three orthogonal imaginary axes, denoted by $\hat{\imath}, \hat{\jmath}$, and $\hat{k}$, which obey Hamilton's rules

$$\begin{aligned} &\hat{\imath}^2 = \hat{\jmath}^2 = \hat{k}^2 = \hat{\imath}\hat{\jmath}\hat{k} = -1, \\ &\hat{\imath}\hat{\jmath} = \hat{k}, \qquad \hat{\jmath}\hat{\imath} = -\hat{k}, \\ &\hat{\jmath}\hat{k} = \hat{\imath}, \qquad \hat{k}\hat{\jmath} = -\hat{\imath}, \quad \text{and} \\ &\hat{k}\hat{\imath} = \hat{\jmath}, \qquad \hat{\imath}\hat{k} = -\hat{\jmath}. \end{aligned} \tag{2.1}$$

It is clear that quaternion multiplication is not commutative. Conventionally, we denote a quaternion $\mathbf{q} = w + x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ by an ordered pair of a real number and a vector, $\mathbf{q} = (w, \mathbf{v}) \in \mathbb{R} \times \mathbb{R}^3$, where $\mathbf{v} = (x, y, z)$. The product of two quaternions

$\mathbf{q}_1$ and $\mathbf{q}_2$ can be written in several forms:

$$
\begin{aligned}
q_1 q_2 &= (w_1 + x_1\hat{i} + y_1\hat{j} + z_1\hat{k})(w_2 + x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) \\
&= w_1w_2 - x_1x_2 - y_1y_2 - z_1z_2 + \\
&\quad (x_1w_2 + w_1x_2 - z_1y_2 + y_1z_2)\hat{i} + \\
&\quad (y_1w_2 + z_1x_2 + w_1y_2 - x_1z_2)\hat{j} + \\
&\quad (z_1w_2 - y_1x_2 + x_1y_2 + w_1z_2)\hat{k},
\end{aligned}
\tag{2.2}
$$

and equivalently

$$
\begin{aligned}
q_1 q_2 &= (w_1, \mathbf{v}_1)(w_2, \mathbf{v}_2) \\
&= (w_1w_2 - \mathbf{v}_1 \cdot \mathbf{v}_2, w_1\mathbf{v}_2 + w_2\mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2).
\end{aligned}
\tag{2.3}
$$

Quaternions form a non-commutative group under multiplication with its identity $\mathbf{1} = (1, 0, 0, 0)$. The inverse of a quaternion $\mathbf{q}$ is $\mathbf{q}^{-1} = (w, -x, -y, -z)/(w^2 + x^2 + y^2 + z^2)$. A quaternion of unit length is called a unit quaternion, which can be considered as a point on the unit hyper-sphere $\mathbb{S}^3$.

From the fundamental theorem presented by Euler, any orientation of a rigid body can be represented as a rotation about a fixed axis $\hat{\mathbf{v}}$ by an angle θ, where $\hat{\mathbf{v}}$ is a 3-dimensional vector of unit length. With a unit quaternion $\mathbf{q} = (\cos\frac{\theta}{2}, \hat{\mathbf{v}}\sin\frac{\theta}{2}) \in \mathbb{S}^3$, we can describe a rotation map $R_{\mathbf{q}} \in \mathbb{SO}(3)$ as follows:

$$
R_{\mathbf{q}}(\mathbf{p}) = \mathbf{q}\mathbf{p}\mathbf{q}^{-1}, \qquad \text{for} \;\; \mathbf{p} \in \mathbb{R}^3. \tag{2.4}
$$

Here, the vector $\mathbf{p} = (x, y, z)$ is interpreted as a purely imaginary quaternion $(0, x, y, z)$. Note that $R_{\mathbf{q}} = R_{-\mathbf{q}}$; that is, two antipodal points, $\mathbf{q}$ and $-\mathbf{q}$ represent the same rotation in $\mathbb{SO}(3)$. Therefore, the mapping between $\mathbb{S}^3$ and $\mathbb{SO}(3)$ is two-to-one.

The tangent vector at a point of a unit quaternion curve $\mathbf{q}(t) \in \mathbb{S}^3$ lies in the tangent space $T_{\mathbf{q}}\mathbb{S}^3$ at the point. Multiplying the tangent vector by the inverse of the quaternion point, we bring every tangent space to coincide with the one at the identity $\mathbf{1} = (1, 0, 0, 0)$. Any tangent vector at the identity is perpendicular to the

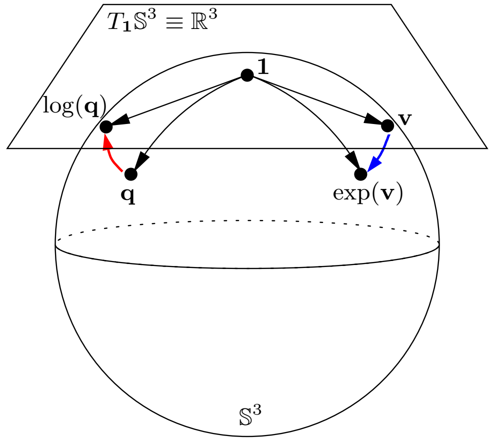

Figure 2.1: Exponential and logarithmic maps

real axis and thus it must be a purely imaginary quaternion, which corresponds to a vector in $\mathbb{R}^3$. In physics terminology, the purely imaginary tangent is identical to the angular velocity $\omega(t) \in \mathbb{R}^3$ of $\mathbf{q}(t)$:

$$\omega(t) = 2\mathbf{q}^{-1}(t)\dot{\mathbf{q}}(t), \tag{2.5}$$

which is measured in the local coordinate frame specified by $\mathbf{q}(t)$. Here, $\frac{\omega(t)}{\|\omega(t)\|}$ is an instantaneous axis of the rotation, and $\|\omega(t)\|$ is the rate of change of the rotation angle about the axis. Since the unit quaternion space is folded by the antipodal equivalence, the angular velocity measured in $\mathbb{S}^3$ is twice as fast as the angular velocity measured in $\mathbb{SO}(3)$. The constant factor 2 in Equation (2.5) keeps consistency between the unit quaternion space and the rotation space.

Exponential and Logarithmic Maps

One of the main connections between vectors and unit quaternions is the exponential mapping. Quaternion exponentiation is defined in the standard way as:

$$\exp(\mathbf{q}) = 1 + \frac{\mathbf{q}}{1!} + \frac{\mathbf{q}}{2!} + \cdots + \frac{\mathbf{q}^n}{n!} + \cdots \tag{2.6}$$

If the real part of $\mathbf{q}$ is zero, then exponential mapping gives a unit quaternion which can be expressed in a closed-form.

$$\exp(\mathbf{q}) = \exp(0, \mathbf{v}) = (\cos \|\mathbf{v}\|, \frac{\mathbf{v}}{\|\mathbf{v}\|} \sin \|\mathbf{v}\|) \in \mathbb{S}^3. \tag{2.7}$$

For simplicity, we often denote $\exp(0, \mathbf{v})$ as $\exp(\mathbf{v})$. This map is onto but not one-to-one. To define its inverse function, we limit the domain such that $\|\mathbf{v}\| < \pi$. Then, the exponential map becomes one-to-one and thus its inverse map $\log : \mathbb{S}^3 \backslash (-1, 0, 0, 0) \rightarrow \mathbb{R}^3$ is well-defined.

$$\log(\mathbf{q}) = \log(w, \mathbf{v}) = \begin{cases} \frac{\pi}{2}\mathbf{v}, & \text{if } w = 0, \\ \frac{\mathbf{v}}{\|\mathbf{v}\|} \tan^{-1} \frac{\|\mathbf{v}\|}{w}, & \text{if } 0 < |w| < 1, \\ \mathbf{0}, & \text{if } w = 1. \end{cases} \tag{2.8}$$

In a geometric viewpoint, the exponential and logarithmic maps provide a correspondence between $\mathbb{S}^3$ and its tangent space $T_{\mathbf{1}}\mathbb{S}^3 \equiv \mathbb{R}^3$ at the identity $\mathbf{1}$ (see Figure 2.1). An element of $T_{\mathbf{1}}\mathbb{S}^3$ can be interpreted as the angular velocity of a certain angular motion at a particular time instance. For any given $\mathbf{q}$ in the neighborhood of the identity, there exists $\mathbf{v} \in T_{\mathbf{1}}\mathbb{S}^3$ which is mapped to $\mathbf{q}$ by exponential mapping, that is, $\mathbf{q} = \exp(\mathbf{v})$. This implies that we can parameterize the neighborhood of the identity with three variables $\mathbf{v} = (x, y, z) \in \mathbb{R}^3$, whereas we need at least four variables to parameterize the entire space $\mathbb{S}^3$ without singularity.

Geodesics

A rigid object starting at orientation $\mathbf{q}_1$ will experience the orientation $\mathbf{q}_2$ when rotating uniformly with constant angular velocity $\mathbf{v} = \log(\mathbf{q}_1^{-1}\mathbf{q}_2)$. The trajectory traversed by the object gives the shortest path on $\mathbb{S}^3$ between $\mathbf{q}_1$ and $\mathbf{q}_2$, and is called a *geodesic* of $\mathbb{S}^3$. In that sense, the *geodesic norm*

$$\text{dist}(\mathbf{q}_1, \mathbf{q}_2) = \| \log(\mathbf{q}_1^{-1}\mathbf{q}_2)\| \tag{2.9}$$

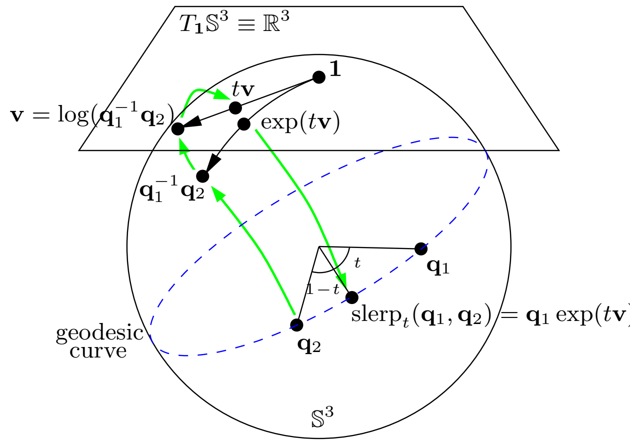

Figure 2.2: Geodesics and spherical linear interpolation

is a natural distance metric in the unit quaternion space. This metric is *bi-invariant*, that is,

$$\begin{aligned} \text{dist}(\mathbf{q}_1, \mathbf{q}_2) &= \text{dist}(\mathbf{a}\mathbf{q}_1, \mathbf{a}\mathbf{q}_2) \\ &= \text{dist}(\mathbf{q}_1\mathbf{b}, \mathbf{q}_2\mathbf{b}) \end{aligned} \tag{2.10}$$

for any $\mathbf{a}, \mathbf{b} \in \mathbb{S}^3$.

The *slerp* (spherical linear interpolation) introduced by Shoemake [88] parameterizes the points on a geodesic curve to compute the in-betweens of two given orientations $\mathbf{q}_1$ and $\mathbf{q}_2$ (see Figure 2.2). That is,

$$\begin{aligned} \text{slerp}_t(\mathbf{q}_1, \mathbf{q}_2) &= \mathbf{q}_1 \exp(t \cdot \log(\mathbf{q}_1^{-1}\mathbf{q}_2)) \\ &= \mathbf{q}_1 (\mathbf{q}_1^{-1}\mathbf{q}_2)^t. \end{aligned} \tag{2.11}$$

$\text{slerp}_t(\mathbf{q}_1, \mathbf{q}_2)$ describes an angular motion on the geodesic between $\mathbf{q}_1$ and $\mathbf{q}_2$ as t changes uniformly from zero to one. This motion has constant angular velocity $\log(\mathbf{q}_1^{-1}\mathbf{q}_2)$.