Chapter 2

Preliminary

Orientations as well as positions are important to describe motions. Quaternions, discovered by Sir William Rowan Hamilton [39], provide a solid base to represent the orientation of a 3-dimensional object. In this chapter, we give a brief introduction to quaternions to explain the geometric and algebraic structures of the unit quaternion space $S^3$.

Quaternion Basics

The four-dimensional space of quaternions is spanned by a real axis and three orthogonal imaginary axes, denoted by $\hat{i}, \hat{j},$ and $\hat{k}$, which obey Hamilton’s rules

\[
\begin{align*}
\hat{i}^2 = \hat{j}^2 = \hat{k}^2 = \hat{i}\hat{j}\hat{k} &= -1, \\
\hat{i}\hat{j} &= \hat{k}, \\
\hat{j}\hat{k} &= \hat{i}, \\
\hat{k}\hat{i} &= \hat{j}, \quad \hat{i}\hat{k} = -\hat{j}, \\
\end{align*}
\]

(2.1)

It is clear that quaternion multiplication is not commutative. Conventionally, we denote a quaternion $q = w + xi + yj + zk$ by an ordered pair of a real number and a vector, $q = (w, v) \in \mathbb{R} \times \mathbb{R}^3$, where $v = (x, y, z)$. The product of two quaternions
\( \mathbf{q}_1 \) and \( \mathbf{q}_2 \) can be written in several forms:

\[
\mathbf{q}_1 \mathbf{q}_2 = (w_1 + x_1 \hat{i} + y_1 \hat{j} + z_1 \hat{k})(w_2 + x_2 \hat{i} + y_2 \hat{j} + z_2 \hat{k})
\]

\[
= w_1 w_2 - x_1 x_2 - y_1 y_2 - z_1 z_2 +
\]

\[
(x_1 w_2 + w_1 x_2 - z_1 y_2 + y_1 z_2) \hat{i} +
\]

\[
(y_1 w_2 + z_1 x_2 + w_1 y_2 - x_1 z_2) \hat{j} +
\]

\[
(z_1 w_2 - y_1 x_2 + x_1 y_2 + w_1 z_2) \hat{k},
\]

and equivalently

\[
\mathbf{q}_1 \mathbf{q}_2 = (w_1, \mathbf{v}_1)(w_2, \mathbf{v}_2)
\]

\[
= (w_1 w_2 - \mathbf{v}_1 \cdot \mathbf{v}_2, \mathbf{v}_1 \mathbf{v}_2 + w_2 \mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2).
\]

Quaternions form a non-commutative group under multiplication with its identity \( \mathbf{1} = (1, 0, 0, 0) \). The inverse of a quaternion \( \mathbf{q} \) is \( \mathbf{q}^{-1} = (w, -x, -y, -z)/(w^2 + x^2 + y^2 + z^2) \). A quaternion of unit length is called a unit quaternion, which can be considered as a point on the unit hyper-sphere \( S^3 \).

From the fundamental theorem presented by Euler, any orientation of a rigid body can be represented as a rotation about a fixed axis \( \hat{\mathbf{v}} \) by an angle \( \theta \), where \( \hat{\mathbf{v}} \) is a 3-dimensional vector of unit length. With a unit quaternion \( \mathbf{q} = (\cos \frac{\theta}{2}, \hat{\mathbf{v}} \sin \frac{\theta}{2}) \in S^3 \), we can describe a rotation map \( R_{\mathbf{q}} \in SO(3) \) as follows:

\[
R_{\mathbf{q}}(\mathbf{p}) = \mathbf{q} \mathbf{p} \mathbf{q}^{-1}, \quad \text{for } \mathbf{p} \in \mathbb{R}^3.
\]

Here, the vector \( \mathbf{p} = (x, y, z) \) is interpreted as a purely imaginary quaternion \((0, x, y, z)\). Note that \( R_{\mathbf{q}} = R_{-\mathbf{q}} \); that is, two antipodal points, \( \mathbf{q} \) and \(-\mathbf{q}\) represent the same rotation in \( SO(3) \). Therefore, the mapping between \( S^3 \) and \( SO(3) \) is two-to-one.

The tangent vector at a point of a unit quaternion curve \( \mathbf{q}(t) \in S^3 \) lies in the tangent space \( T_{\mathbf{q}}S^3 \) at the point. Multiplying the tangent vector by the inverse of the quaternion point, we bring every tangent space to coincide with the one at the identity \( \mathbf{1} = (1, 0, 0, 0) \). Any tangent vector at the identity is perpendicular to the
real axis and thus it must be a purely imaginary quaternion, which corresponds to a vector in $\mathbb{R}^3$. In physics terminology, the purely imaginary tangent is identical to the angular velocity $\omega(t) \in \mathbb{R}^3$ of $q(t)$:

$$\omega(t) = 2q^{-1}(t)\dot{q}(t),$$

(2.5)

which is measured in the local coordinate frame specified by $q(t)$. Here, $\frac{\omega(t)}{\|\omega(t)\|}$ is an instantaneous axis of the rotation, and $\|\omega(t)\|$ is the rate of change of the rotation angle about the axis. Since the unit quaternion space is folded by the antipodal equivalence, the angular velocity measured in $S^3$ is twice as fast as the angular velocity measured in $SO(3)$. The constant factor 2 in Equation (2.5) keeps consistency between the unit quaternion space and the rotation space.

**Exponential and Logarithmic Maps**

One of the main connections between vectors and unit quaternions is the exponential mapping. Quaternion exponentiation is defined in the standard way as:

$$\exp(q) = 1 + \frac{q}{1!} + \frac{q^2}{2!} + \cdots + \frac{q^n}{n!} + \cdots$$

(2.6)
If the real part of \( q \) is zero, then exponential mapping gives a unit quaternion which can be expressed in a closed-form.

\[
\exp(q) = \exp(0, \mathbf{v}) = (\cos \| \mathbf{v} \|, \frac{\mathbf{v}}{\| \mathbf{v} \|} \sin \| \mathbf{v} \|) \in S^3.
\] (2.7)

For simplicity, we often denote \( \exp(0, \mathbf{v}) \) as \( \exp(\mathbf{v}) \). This map is onto but not one-to-one. To define its inverse function, we limit the domain such that \( \| \mathbf{v} \| < \pi \). Then, the exponential map becomes one-to-one and thus its inverse map \( \log : S^3 \setminus (-1, 0, 0, 0) \to \mathbb{R}^3 \) is well-defined.

\[
\log(q) = \log(w, \mathbf{v}) = \begin{cases} 
\frac{\mathbf{v}}{\| \mathbf{v} \|}, & \text{if } w = 0, \\
\frac{\mathbf{v}}{\| \mathbf{v} \|} \tan^{-1} \frac{\| \mathbf{v} \|}{w}, & \text{if } 0 < |w| < 1, \\
0, & \text{if } w = 1.
\end{cases}
\] (2.8)

In a geometric viewpoint, the exponential and logarithmic maps provide a correspondence between \( S^3 \) and its tangent space \( T_1 S^3 \equiv \mathbb{R}^3 \) at the identity \( \mathbf{1} \) (see Figure 2.1). An element of \( T_1 S^3 \) can be interpreted as the angular velocity of a certain angular motion at a particular time instance. For any given \( q \) in the neighborhood of the identity, there exists \( \mathbf{v} \in T_1 S^3 \) which is mapped to \( q \) by exponential mapping, that is, \( q = \exp(\mathbf{v}) \). This implies that we can parameterize the neighborhood of the identity with three variables \( \mathbf{v} = (x, y, z) \in \mathbb{R}^3 \), whereas we need at least four variables to parameterize the entire space \( S^3 \) without singularity.

**Geodesics**

A rigid object starting at orientation \( q_1 \) will experience the orientation \( q_2 \) when rotating uniformly with constant angular velocity \( \mathbf{v} = \log(q_1^{-1} q_2) \). The trajectory traversed by the object gives the shortest path on \( S^3 \) between \( q_1 \) and \( q_2 \), and is called a *geodesic* of \( S^3 \). In that sense, the *geodesic norm*

\[
\text{dist}(q_1, q_2) = \| \log(q_1^{-1} q_2) \| \quad (2.9)
\]
Figure 2.2: Geodesics and spherical linear interpolation

is a natural distance metric in the unit quaternion space. This metric is bi-invariant, that is,

\[
\text{dist}(q_1, q_2) = \text{dist}(aq_1, aq_2) = \text{dist}(q_1b, q_2b)
\]

(2.10)

for any \(a, b \in S^3\).

The slerp (spherical linear interpolation) introduced by Shoemake [88] parameterizes the points on a geodesic curve to compute the in-betweens of two given orientations \(q_1\) and \(q_2\) (see Figure 2.2). That is,

\[
\text{slerp}_t(q_1, q_2) = q_1 \exp(t \cdot \log(q_1^{-1}q_2)) = q_1(q_1^{-1}q_2)^t.
\]

(2.11)

\(\text{slerp}_t(q_1, q_2)\) describes an angular motion on the geodesic between \(q_1\) and \(q_2\) as \(t\) changes uniformly from zero to one. This motion has constant angular velocity \(\log(q_1^{-1}q_2)\).